**Inequality 3**

1. Prove that $(a + b) (b + c) (c + a) \geq 8abc$ is true for all positive numbers a, b, and c with equality only if $a = b = c$.

 Let $a , b,c$ be positive numbers.

 $\left(\sqrt{a}-\sqrt{b}\right)^{2}\geq 0 ⟺ a + b\geq 2\sqrt{ab}$ with equality holds only if $a = b$

 Similarly, $b + c\geq 2\sqrt{bc}$ with equality holds only if $b = c$

 $c + a\geq 2\sqrt{ca}$ with equality holds only if $c = a$

 Hence $\left(a + b\right)\left(b + c\right)\left(c + a\right)\geq \left(2\sqrt{ab}\right)\left(2\sqrt{bc}\right)\left(2\sqrt{ca}\right)=8abc$

 with equality holds only if $a = b=c$.

2. Solve $\frac{x\left(x+2\right)}{x-1}\leq 0$.

 $\frac{x\left(x+2\right)}{x-1}\leq 0⟹\left\{\begin{array}{c}x\left(x+2\right)\geq 0\\x-1<0\end{array}\right. or \left\{\begin{array}{c}x\left(x+2\right)\leq 0\\x-1>0\end{array}\right.$

 $⟹\left\{\begin{array}{c}-2\geq x or x\geq 0\\x<1\end{array}\right. or \left\{\begin{array}{c}-2\leq x\leq 0\\x>1\end{array}\right.$

 $⟹\left(-2\geq x or 0\leq x<1\right) or no solution$

 $⟹-2\geq x or 0\leq x<1$

3. Solve $\frac{9}{1-x}\leq \frac{7x+5}{x+3}$.

 $\frac{7x+5}{x+3}+\frac{9}{x-1}\geq 0$

 $\frac{7x^{2}+7x+22}{\left(x+3\right)\left(x-1\right)}\geq 0…(\*)$

 **Method 1**

 Consider: $f\left(x\right)=7x^{2}+7x+22$. Since $∆ of f(x)=7^{2}-4\left(7\right)\left(22\right)=-576<0$

 The curve $f\left(x\right)$ cannot cut the axis and is above the x-axis since $a=7>0$.

 $∴7x^{2}+7x+22>0$ for all $x\in R$.

 **Method 2**

 $f\left(x\right)=a\left(x+\frac{b}{2a}\right)^{2}-\frac{∆}{4a}=7\left(x+\frac{7}{14}\right)^{2}-\frac{-576}{28}=7\left(x+\frac{1}{2}\right)^{2}+\frac{144}{7}>0+\frac{144}{7}>0$

 Hence from $(\*)$,$\left(x+3\right)\left(x-1\right)>0⟹x<-3 or x>1$.

4. Solve $|2x-3|<|x-1|+|x-2|$ for $x$.

$|2x-3|<|x-1|+|x-2|⟹|\left(x-1\right)+\left(x-2\right)|<|x-1|+|x-2|$

 Since both sides are positive, we square both sides,

 $\left(x-1\right)^{2}+2\left(x-1\right)\left(x-2\right)+\left(x-2\right)^{2}<\left(x-1\right)^{2}+2\left|\left(x-1\right)\left(x-2\right)\right|+\left(x-2\right)^{2}$

 $\left(x-1\right)\left(x-2\right)<\left|\left(x-1\right)\left(x-2\right)\right|$

 This is a strict inequality, we only have $\left(x-1\right)\left(x-2\right)<0⟹1<x<2$

5. (a) Find the solution of the general symmetric inequality: $\left|x+a\right|+\left|x-a\right|\leq b, b\geq 0$

 (b) Hence find $x$ where $\left|x-1\right|+\left|x-2\right|\leq 4$.

 (a) $\left|x+a\right|+\left|x-a\right|\leq b, b\geq 0…(1)$ By triangular inequalities $\left|x+a\right|+\left|x-a\right|\geq \left|\left(x+a\right)+\left(x-a\right)\right|=2\left|x\right|$ (1) becomes: $\left|x\right|\leq \frac{b}{2}⟺-\frac{b}{2}\leq x\leq \frac{b}{2}$ Note also, $\left|x+a\right|+\left|x-a\right|=\left|x+a\right|+\left|a-x\right|\geq \left|\left(x+a\right)+\left(a-x\right)\right|=2\left|a\right|$ (1) becomes: $\left|a\right|\leq \frac{b}{2}⟺-\frac{b}{2}\leq a\leq \frac{b}{2}$ In general, (1) has solution: $-\frac{b}{2}\leq x\leq \frac{b}{2}$ , if $-\frac{b}{2}\leq a\leq \frac{b}{2}$.

 (b) Now we go back to $\left|x-1\right|+\left|x-2\right|\leq 4…(2)$ (2) obviously is not the same as the symmetric form as in (1). Put $u=x-1.5$, we get $\left|u+0.5\right|+\left|u-0.5\right|\leq 4$ ($-1.5$ is the mid point of $-1$ and $-2$) Apply previous result, $-2\leq u\leq 2⟹-2\leq x-1.5\leq 2⟹-0.5\leq x\leq 3.5$

6. Solve $\left|x-\frac{1}{x}\right|<4$.

 We assume $x$ is real.

 Since both sides are positive, squaring gives $\left(x-\frac{1}{x}\right)^{2}<16$

 Since $x\ne 0$, multiple both sides by $x^{2}>0$, $\left(x^{2}-1\right)^{2}<16x^{2}$

 Hence $\left(x^{2}-1\right)^{2}-\left(4x\right)^{2}<0$

 $⟹\left(x^{2}-4x-1\right)\left(x^{2}+4x-1\right)<0$

 $⟹\left\{\begin{array}{c}x^{2}-4x-1>0\\x^{2}+4x-1<0\end{array}\right. or \left\{\begin{array}{c}x^{2}-4x-1<0\\x^{2}+4x-1>0\end{array}\right.$

 $⟹\left\{\begin{array}{c}\left[x-\left(2-\sqrt{5}\right)\right]\left[x-\left(2+\sqrt{5}\right)\right]>0\\\left[x-\left(-2-\sqrt{5}\right)\right]\left[x-\left(-2+\sqrt{5}\right)\right]<0\end{array}\right. or \left\{\begin{array}{c}\left[x-\left(2-\sqrt{5}\right)\right]\left[x-\left(2+\sqrt{5}\right)\right]<0\\\left[x-\left(-2-\sqrt{5}\right)\right]\left[x-\left(-2+\sqrt{5}\right)\right]>0\end{array}\right.$

 $⟹\left\{\begin{array}{c}2-\sqrt{5}>x or x>2+\sqrt{5} \\-2-\sqrt{5}<x<-2+\sqrt{5}\end{array}\right. or \left\{\begin{array}{c}2-\sqrt{5}<x<2+\sqrt{5}\\-2-\sqrt{5}>x or x>\sqrt{5}-2\end{array}\right.$

 $⟹-2-\sqrt{5}<x<2-\sqrt{5} or \sqrt{5}-2<x<2+\sqrt{5}$

7. Prove by mathematical induction the inequality $5^{n}\geq n^{5}$ for all $n\geq 5$.

 Let $P\left(n\right):5^{n}\geq n^{5}$

 For $\left(5\right)$ , $5^{5}\geq 5^{5}$ is obviously true.

 Assume $P\left(k\right)$ is true for some $k\in N,k\geq 5 $, that is $5^{k}\geq k^{5}…(1)$

 For $P\left(k+1\right)$,

 $5^{k+1}=5∙5^{k}\geq 5k^{5}$ , by (1)

 $=k^{5}+k^{5}+k^{5}+k^{5}+k^{5}$

 $=k^{5}+kk^{4}+k^{2}k^{3}+k^{3}k^{2}+k^{4}k$ ,$ k\geq 5$

 $\geq k^{5}+5k^{4}+25k^{3}+125k^{2}+624k+k$ , $k\geq 5$

 $\geq k^{5}+5k^{4}+10k^{3}+10k^{2}+5k+1$ ,$ k\geq 5$

 $=\left(k+1\right)^{5}$

 $∴P\left(k+1\right)$ is true.

 By the principle of mathematical induction, $P\left(n\right)$ is true for all $n\in N, n\geq 5.$

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